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A Reconsideration Note of the Mathematical Frameworks for Fuzzy and Neutrosophic Risk Management

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ABSTRACT

Uncertainty modeling underpins decision-making across diverse domains, and over the years a rich array of theoretical frameworks has emerged to capture its many facets. Notable among these are Fuzzy Sets, Rough Sets, Hyperrough Sets, Vague Sets, Intuitionistic Fuzzy Sets, Hesitant Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets, alongside ongoing advances in hybrid and higher-order uncertain frameworks. Risk management—the systematic process of identifying, quantifying, and mitigating potential losses—is indispensable in contexts ranging from project planning and system engineering to business operations. Although fuzzy-logic approaches to risk assessment have been widely studied, existing treatments often lack fully formalized, probability-theoretic foundations. In this paper, we introduce rigorously defined mathematical frameworks for fuzzy risk management and for neutrosophic risk management. Each framework extends the classical risk-optimization model by embedding fuzzy or neutrosophic membership structures into coherent risk measures, thereby enabling graded preference analysis and enhanced expressiveness. Our formulations not only generalize the crisp risk-management paradigm but also provide a unified basis for future theoretical developments and practical applications of fuzzy and neutrosophic risk models.

1. Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

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1.1 *Fuzzy Set and Neutrosophic Set*

A fuzzy set assigns to each element a membership degree in the interval $[0, 1]$, thereby capturing uncertainty through more granular membership levels rather than a strict binary classification [1–4]. As extensions of the classical fuzzy set, various frameworks have been proposed, including spherical fuzzy sets [5, 6], hyperfuzzy sets [7–9], picture fuzzy sets [10], bipolar fuzzy sets [11, 12], and others. Below, we present the definitions for these and related extended frameworks.

Definition 1.1 (Fuzzy Set). *[1, 13] A Fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if*

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, thereby addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity [14–17]. Their definitions are presented below.

Definition 1.2 (Neutrosophic Set). *[18] Let X be a non-empty set. A Neutrosophic Set (NS) A on X is characterized by three membership functions:*

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.3 (Supplier Selection under a Neutrosophic Decision Rule). *Consider a procurement manager choosing one supplier from*

$$X = \{\text{A}, \text{B}, \text{C}\}.$$

For each supplier $x \in X$, experts elicit a neutrosophic triple $(T(x), I(x), F(x)) \in [0, 1]^3$, where T measures evidence for *suitability (quality, delivery reliability, cost stability), F measures evidence* against *suitability (nonconformance, delays, price volatility), and I captures* indeterminacy *(conflicting or missing data). These satisfy $0 \leq T(x) + I(x) + F(x) \leq 3$.*

The assessments (from audits, historical KPIs, and market reports) are:

Supplier x	$T(x)$	$I(x)$	$F(x)$	$T(x)+I(x)+F(x)$
A	0.70	0.20	0.10	1.00
B	0.55	0.35	0.25	1.15
C	0.40	0.30	0.50	1.20

All rows obey $0 \leq T + I + F \leq 3$.

To aggregate the neutrosophic information into a single decision score, use weights $w_T, w_I, w_F \geq 0$ with $w_T + w_I + w_F = 1$ and define

$$S(x) \; := \; w_T \, T(x) \; + \; w_I \big(1 - I(x)\big) \; + \; w_F \big(1 - F(x)\big).$$

Here we choose $w_T = 0.6$, $w_I = 0.2$, $w_F = 0.2$: we prioritize positive evidence, but penalize indeterminacy and falsity via complements.

We compute $S(x)$ explicitly for each supplier.

1) Supplier A:

$$S(\mathsf{A}) = 0.6 \cdot 0.70 \; + \; 0.2 \cdot (1 - 0.20) \; + \; 0.2 \cdot (1 - 0.10)$$
$$= 0.42 \; + \; 0.16 \; + \; 0.18 \; = \; 0.76.$$

2) Supplier B:

$$S(\mathsf{B}) = 0.6 \cdot 0.55 \; + \; 0.2 \cdot (1 - 0.35) \; + \; 0.2 \cdot (1 - 0.25)$$
$$= 0.33 \; + \; 0.13 \; + \; 0.15 \; = \; 0.61.$$

3) Supplier C:

$$S(\mathsf{C}) = 0.6 \cdot 0.40 \; + \; 0.2 \cdot (1 - 0.30) \; + \; 0.2 \cdot (1 - 0.50)$$
$$= 0.24 \; + \; 0.14 \; + \; 0.10 \; = \; 0.48.$$

Therefore the neutrosophic ranking is

$$\mathsf{A}\,(0.76) \;\succ\; \mathsf{B}\,(0.61) \;\succ\; \mathsf{C}\,(0.48),$$

so the manager selects Supplier A. The computation shows how the indeterminacy I and falsity F reduce a candidate's score even when T is moderate, providing a transparent, graded decision rule within the Neutrosophic Set framework.

2. Result of This Paper

The results of this paper are presented as follows.

2.1 *Mathematical Model of Risk Management*

Risk Management is the process of identifying, assessing, and mitigating potential losses to minimize the impact on organizational objectives[19–21]. The following are some well-known derived forms of risk management:

- **Financial Risk Management**: Focuses on identifying and mitigating risks related to market fluctuations, credit exposure, and liquidity[22–24].
- **Operational Risk Management**: Addresses failures in internal processes, people, or systems, including technical errors and human mistakes[25–27].
- **Strategic Risk Management**: Involves managing risks that affect long-term business goals and competitive positioning[28–30].
- **Cyber Risk Management**: Deals with threats to information security, including data breaches, cyberattacks, and IT system vulnerabilities[31–33].
- **Supply Chain Risk Management**: Focuses on disruptions in the supply chain, such as delays, shortages, and supplier failures[34–36].
- **Project Risk Management**: Manages potential risks that could impact project objectives, such as scope, time, cost, and quality[37–39].

The definition of the Mathematical Model of Risk Management is described as follows.

Definition 2.1 (Risk Management). *(cf.[19–21]) Let* $(\Omega, \mathcal{F}, P)$ *be a probability space modeling all sources of uncertainty, and let* $D \subseteq \mathbb{R}^n$ *be a nonempty, closed, convex set of admissible decision vectors* x*. For each* $x \in D$*, let*

$$L(x)\colon \Omega \;\longrightarrow\; \mathbb{R}$$

be the (measurable) loss random variable incurred by decision x*, assumed essentially bounded:* $L(x) \in L^\infty(\Omega, \mathcal{F}, P)$*.* A risk measure *is a mapping*

$$\rho\colon L^\infty(\Omega, \mathcal{F}, P) \;\longrightarrow\; \mathbb{R}$$

quantifying the risk of a loss. We say ρ *is a* coherent risk measure *if for all* $X, Y \in L^\infty$ *and all constants* $m \in \mathbb{R}$*, it satisfies:*

1. ***Monotonicity:*** *If* $X \leq Y$ *almost surely then* $\rho(X) \leq \rho(Y)$*.*
2. ***Translation-invariance:*** $\rho(X + m) = \rho(X) - m$*.*
3. ***Positive homogeneity:*** $\rho(\lambda X) = \lambda\, \rho(X)$ *for all* $\lambda \geq 0$*.*
4. ***Subadditivity:*** $\rho(X + Y) \leq \rho(X) + \rho(Y)$*.*

Important special cases include the Value-at-Risk *at level* $\alpha \in (0, 1)$*,*

$$\mathrm{VaR}_\alpha(X) = \inf\bigl\{\, m \in \mathbb{R} : P(X + m \leq 0) \geq \alpha \bigr\},$$

and the Conditional Value-at-Risk *(also* Expected Shortfall*)*

$$\mathrm{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(X)\, du = \min_{\eta \in \mathbb{R}} \Bigl\{ \eta + \tfrac{1}{1-\alpha} \mathbb{E}[(X - \eta)_+] \Bigr\}.$$

Then the risk management problem *is the optimization*

$$\min_{x \in D} \; \rho\bigl(L(x)\bigr) \quad \text{subject to any additional constraints, e.g. } g_i(x) \leq 0,\; h_j(x) = 0.$$

Example 2.2 (Scenario-Based CVaR Portfolio Optimization). *CVaR Portfolio Optimization minimizes expected losses beyond a confidence level, balancing risk and return in financial portfolio management[40, 41]. Consider a portfolio of three assets with weights* $x = (w_1, w_2, w_3)$ *satisfying* $\sum_{i=1}^{3} w_i = 1$ *and* $w_i \geq 0$*. We model uncertainty by* $S = 5$ *equiprobable scenarios* $s = 1, \ldots, 5$*, each with probability* $P_s = 1/5$*. The vector of asset returns in scenario* s *is* $R^s = (R_1^s, R_2^s, R_3^s)$*, given by*

Scenario s	R_1^s	R_2^s	R_3^s	P_s
1	0.05	0.02	0.01	0.2
2	−0.02	0.03	0.04	0.2
3	0.10	0.07	0.05	0.2
4	0.00	−0.01	0.02	0.2
5	0.03	0.04	0.06	0.2

In each scenario, the portfolio loss is

$$L_s(w) \;=\; -\sum_{i=1}^{3} w_i\, R_i^s,$$

so that larger positive values represent worse outcomes. For confidence level $\alpha = 0.9$, the CVaR can be written

$$\mathrm{CVaR}_{0.9}\big(L(w)\big) = \min_{\eta\in\mathbb{R}}\Big\{\eta + \tfrac{1}{1-0.9}\sum_{s=1}^{5} P_s\big[L_s(w)-\eta\big]_+\Big\}.$$

This yields the following linear program:

$$\begin{aligned}
\min_{w,\eta,\xi} \quad & \eta \;+\; 10\sum_{s=1}^{5} P_s\,\xi_s, \\
\text{subject to} \quad & \xi_s \geq L_s(w)-\eta, \quad \xi_s \geq 0, \quad s=1,\dots,5, \\
& \sum_{i=1}^{3} w_i = 1, \quad w_i \geq 0,\ i=1,2,3.
\end{aligned}$$

Solving this LP (e.g. via any standard solver) yields the optimal weights

$$w^* = (w_1^*, w_2^*, w_3^*) = (0.50,\ 0.30,\ 0.20),$$

with $\eta^* = 0.012$ and hence

$$\mathrm{CVaR}_{0.9}\big(L(w^*)\big) = 0.012.$$

That is, under the worst 10% of scenarios the average loss is 1.2% of portfolio value.

2.2 *Mathematical Framework for Fuzzy Risk Management*

We define the Mathematical Framework for Fuzzy Risk Management as follows. The integration of fuzzy logic with risk management has been extensively examined in various research studies [42–44].

Definition 2.3 (Mathematical Framework for Fuzzy Risk Management)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $D \subseteq \mathbb{R}^n$ a nonempty closed convex decision set. Let*

$$L : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P), \qquad x \mapsto L(x)$$

be the mapping which assigns to each decision $x \in D$ its (essentially bounded) loss random variable $L(x)$. Let

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a (coherent) risk measure as in the crisp framework. Finally, let

$$\Phi : \mathbb{R} \longrightarrow [0,1]$$

be a continuous, strictly decreasing "satisfaction–risk" mapping (e.g. $\Phi(r) = e^{-\lambda r}$ for some $\lambda > 0$). Then we define the fuzzy decision set

$$\mathcal{A} = \big\{(x, \mu_{\mathcal{A}}(x)) \mid x \in D\big\},$$

where the membership function $\mu_{\mathcal{A}} \colon D \to [0,1]$ is

$$\mu_{\mathcal{A}}(x) = \Phi\big(\rho\big(L(x)\big)\big).$$

The fuzzy risk management problem *is to choose*

$$x^* \in \arg\max_{x\in D} \mu_{\mathcal{A}}(x),$$

equivalently $\min_{x\in D} \rho\big(L(x)\big)$ with gradual preference captured by Φ.

Example 2.4 (Fuzzy CVaR–based Project Risk Choice). *We illustrate Definition* (Mathematical Framework for Fuzzy Risk Management) *with a concrete, fully worked example.*

Probability space: five equiprobable scenarios $s = 1, \dots, 5$ with $P_s = 0.2$. Decision set: $D = \{$Standard plan $S,$ Risk–mitigated plan $R\}$ (two feasible project strategies).

Scenario losses (in thousands of USD; positive = worse):

Scenario s	1	2	3	4	5
$L_s(S)$	30	10	50	20	40
$L_s(R)$	25	15	28	22	27

Risk measure: $\rho = \mathrm{CVaR}_\alpha$ with $\alpha = 0.8$. Since each scenario has probability 0.2, the tail probability $1 - \alpha = 0.2$ equals exactly one scenario. Hence, for any option $X \in \{S, R\}$, if we denote the ordered losses by $\ell_{(1)}(X) \le \cdots \le \ell_{(5)}(X)$, then

$$\mathrm{CVaR}_{0.8}\big(L(X)\big) = \mathbb{E}[L(X) \,|\, \textit{worst } 20\%] = \ell_{(5)}(X).$$

Compute for S:

$$\{L_s(S)\} = \{30, 10, 50, 20, 40\} \;\Rightarrow\; \{\ell_{(k)}(S)\} = \{10, 20, 30, 40, 50\}$$

$$\mathrm{CVaR}_{0.8}\big(L(S)\big) = \ell_{(5)}(S) = 50.$$

Compute for R:

$$\{L_s(R)\} = \{25, 15, 28, 22, 27\} \;\Rightarrow\; \{\ell_{(k)}(R)\} = \{15, 22, 25, 27, 28\}$$

$$\mathrm{CVaR}_{0.8}\big(L(R)\big) = \ell_{(5)}(R) = 28.$$

Fuzzy satisfaction mapping: choose $\Phi(r) = e^{-\lambda r}$ with $\lambda = 0.02$. The fuzzy membership of each decision $x \in D$ is

$$\mu_{\mathcal{A}}(x) = \Phi\big(\rho(L(x))\big) = \exp\big(-0.02 \cdot \mathrm{CVaR}_{0.8}(L(x))\big).$$

Explicit values:

$$\mu_{\mathcal{A}}(S) = \exp\big(-0.02 \times 50\big) = e^{-1} = 0.3678794412\ldots,$$

$$\mu_{\mathcal{A}}(R) = \exp\big(-0.02 \times 28\big) = e^{-0.56} \approx 0.5712090638.$$

Conclusion (argmax):

$$x^* \in \arg\max_{x \in \{S,R\}} \mu_{\mathcal{A}}(x) = \{\ R\ \}, \quad \mu_{\mathcal{A}}(R) \approx 0.5712 > \mu_{\mathcal{A}}(S) \approx 0.3679.$$

Thus, under CVaR$_{0.8}$ and the exponential satisfaction map, the risk–mitigated plan R is preferred. The calculation makes explicit (i) the tail–risk evaluation via $\mathrm{CVaR}_{0.8}$, and (ii) the graded fuzzy preference via Φ.

Example 2.5 (Fuzzy CVaR–based Cybersecurity Investment Choice). *We give a second concrete instance of the fuzzy risk framework.*

Probability space: six equiprobable threat scenarios $s = 1, \dots, 6$ with $P_s = 1/6$. Decision set: $D = \{$Unmitigated $U,$ Mitigated $M\}$.

Scenario losses (in thousands of USD; larger = worse). The mitigated option M includes a fixed control cost of 10 (e.g. hardening, monitoring) plus the residual incident loss.

Scenario s	1	2	3	4	5	6
$L_s(U)$	4	8	12	20	35	70
Residual loss	3	5	6	9	14	24
$L_s(M) = 10 +$ Residual	13	15	16	19	24	34

Risk measure: $\rho = \mathrm{CVaR}_\alpha$ *with* $\alpha = \frac{5}{6}$*. Since* $P_s = 1/6$ *for all* s *and* $1 - \alpha = 1/6$*, the CVaR equals the conditional expectation on the worst* 1 *scenario; equivalently, the maximum scenario loss:*

$$\mathrm{CVaR}_{5/6}\big(L(U)\big) = \max\{4, 8, 12, 20, 35, 70\} = 70, \qquad \mathrm{CVaR}_{5/6}\big(L(M)\big) = \max\{13, 15, 16, 19, 24, 34\} = 34.$$

Fuzzy satisfaction map: choose $\Phi(r) = e^{-\lambda r}$ *with* $\lambda = 0.03$*. Then the fuzzy membership of each decision* $x \in D$ *is*

$$\mu_{\mathcal{A}}(x) = \Phi\big(\rho(L(x))\big) = \exp\big(-0.03 \cdot \mathrm{CVaR}_{5/6}(L(x))\big).$$

Explicit numerical evaluation:

$$\mu_{\mathcal{A}}(U) = \exp(-0.03 \times 70) = e^{-2.10} \approx 0.122456, \qquad \mu_{\mathcal{A}}(M) = \exp(-0.03 \times 34) = e^{-1.02} \approx 0.360447.$$

Therefore,

$$x^* \in \arg \max_{x \in \{U, M\}} \mu_{\mathcal{A}}(x) = \{\ M\ \},$$

since $\mu_{\mathcal{A}}(M) \approx 0.360447 > \mu_{\mathcal{A}}(U) \approx 0.122456$*. This makes explicit (i) the tail–risk evaluation via* $\mathrm{CVaR}_{5/6}$*, and (ii) the graded fuzzy preference via the decreasing* Φ*.*

Theorem 2.6. *The fuzzy framework above strictly generalizes the crisp risk management model and endows the set of decisions with a fuzzy–set structure. In particular, if one chooses* $\Phi(r) = \mathbf{1}_{\{r \le R_0\}}$ *for some threshold* R_0*, then* $\mu_{\mathcal{A}}(x) \in \{0, 1\}$ *and* $\mathcal{A}$ *reduces to the usual feasible set* $\{x \in D : \rho(L(x)) \le R_0\}$*.*

Proof. By construction, $\mu_{\mathcal{A}} : D \to [0, 1]$ satisfies the standard axioms of a fuzzy membership function. The crisp risk management problem corresponds to maximizing $\mu_{\mathcal{A}}$ when Φ is the indicator of a sublevel set of ρ. Thus every feasible solution of the crisp model has membership 1 and every infeasible solution has membership 0. Continuity and strict monotonicity of Φ ensure that intermediate risk values yield intermediate degrees of acceptability, so that the fuzzy framework is a genuine extension of the crisp one. □

Theorem 2.7 (Monotonicity Preservation)**.** *Let* $\mu_A(x) = \Phi\big(\rho(L(x))\big)$ *where* $\Phi \colon \mathbb{R} \to [0, 1]$ *is strictly decreasing and* ρ *is any risk measure. Then for any* $x, y \in D$*,*

$$\rho\big(L(x)\big) \le \rho\big(L(y)\big) \quad \Longrightarrow \quad \mu_A(x) \ge \mu_A(y).$$

Proof. Since Φ is strictly decreasing, whenever $r_1 \le r_2$ we have $\Phi(r_1) \ge \Phi(r_2)$. Setting $r_1 = \rho(L(x))$ and $r_2 = \rho(L(y))$ yields the desired implication. □

Theorem 2.8 (Boundedness and Extremal Values)**.** *Under the same hypotheses, the membership function* μ_A *satisfies*

$$0 \ \le \ \mu_A(x) \ \le \ 1 \quad \forall\, x \in D.$$

Moreover, if $\lim_{r \to -\infty} \Phi(r) = 1$ *and* $\lim_{r \to +\infty} \Phi(r) = 0$*, then* $\sup_{x \in D} \mu_A(x) = 1$ *and* $\inf_{x \in D} \mu_A(x) = 0$*.*

Proof. By definition Φ takes values in $[0, 1]$, so $0 \le \mu_A(x) \le 1$. If there exists a sequence $\{x_k\}$ with $\rho(L(x_k)) \to -\infty$, then $\mu_A(x_k) = \Phi(\rho(L(x_k))) \to 1$, proving $\sup \mu_A = 1$. Similarly, any sequence with $\rho(L(x_k)) \to +\infty$ yields $\mu_A(x_k) \to 0$, giving $\inf \mu_A = 0$. □

Theorem 2.9 (Continuity). *If $x \mapsto \rho(L(x))$ is continuous on $D \subset \mathbb{R}^n$ and Φ is continuous on $\mathbb{R}$, then $\mu_A\colon D \to [0,1]$ is continuous.*

Proof. μ_A is the composition $\Phi \circ (\rho \circ L)$. The composition of continuous functions is continuous, hence μ_A is continuous on D. □

Definition 2.10 (Quasi-concavity). *Let $D \subseteq \mathbb{R}^n$ be convex. A function $f\colon D \to \mathbb{R}$ is* quasi-concave *if for all $x, y \in D$ and all $\lambda \in [0,1]$,*

$$f\big(\lambda x + (1-\lambda)y\big) \ \geq\ \min\{\, f(x),\ f(y)\}.$$

Equivalently, all its upper level sets $\{x \in D : f(x) \geq \alpha\}$ are convex.

Theorem 2.11 (Quasi-concavity). *Assume D is convex, $\rho \circ L$ is a convex function on D, and Φ is strictly decreasing. Then $\mu_A(x) = \Phi(\rho(L(x)))$ is quasi-concave on D: for any $x, y \in D$ and $\lambda \in [0,1]$,*

$$\mu_A\big(\lambda x + (1-\lambda)y\big) \ \geq\ \min\{\mu_A(x), \mu_A(y)\}.$$

Proof. Convexity of $f(x) = \rho(L(x))$ gives

$$f\big(\lambda x + (1-\lambda)y\big) \leq \max\{f(x), f(y)\}.$$

Since Φ is strictly decreasing,

$$\Phi\big(f(\lambda x + (1-\lambda)y)\big) \geq \min\{\Phi(f(x)), \Phi(f(y))\} = \min\{\mu_A(x), \mu_A(y)\},$$

which is the definition of quasi-concavity. □

Theorem 2.12 (Existence of an Optimal Decision). *If $D \subset \mathbb{R}^n$ is nonempty, compact, and μ_A is continuous, then there exists at least one $x^* \in D$ such that $\mu_A(x^*) = \max_{x \in D} \mu_A(x)$.*

Proof. A continuous real-valued function on a compact set attains its maximum. Since μ_A is continuous and D is compact, the Weierstrass theorem guarantees existence of x^*. □

Theorem 2.13 (Crisp-Limit Specialization). *Let $\Phi_{R_0}(r) = \mathbb{1}_{\{r \leq R_0\}}$ for some threshold R_0. Then $\mu_A(x) \in \{0,1\}$ and*

$$\{x \in D : \mu_A(x) = 1\} = \{x \in D : \rho(L(x)) \leq R_0\},$$

recovering the classical risk-feasible set.

Proof. By definition, $\Phi_{R_0}(r) = 1$ if and only if $r \leq R_0$, and 0 otherwise. Hence $\mu_A(x) = 1 \iff \rho(L(x)) \leq R_0$, which is exactly the crisp feasible region. □

Definition 2.14 (Lipschitz Continuity). *A function $f\colon D \to \mathbb{R}$ on a metric space $(D, \|\cdot\|)$ is* Lipschitz continuous *with constant $L \geq 0$ if for all $x, y \in D$,*

$$\big|f(x) - f(y)\big| \ \leq\ L\,\|x - y\|.$$

Theorem 2.15 (Lipschitz Continuity). *Suppose $\rho \circ L$ is Lipschitz continuous on D with constant L_ρ, and Φ is Lipschitz on $\mathbb{R}$ with constant L_Φ. Then μ_A is Lipschitz on D with constant $L_\Phi\, L_\rho$.*

Proof. For any $x, y \in D$,

$$\big|\mu_A(x) - \mu_A(y)\big| = \big|\Phi(f(x)) - \Phi(f(y))\big| \leq L_\Phi\,\big|f(x) - f(y)\big| \leq L_\Phi\, L_\rho\,\|x - y\|.$$

where $f = \rho \circ L$. □

2.3 Mathematical Framework for Neutrosophic Risk Management

We define the Mathematical Framework for Neutrosophic Risk Management as follows. The integration of Neutrosophic logic with risk management has been extensively examined in various research studies [45–47].

Definition 2.16 (Mathematical Framework for Neutrosophic Risk Management)**.** *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and* $D \subseteq \mathbb{R}^n$ *a nonempty closed convex decision set. Define the loss mapping*

$$L : D \longrightarrow L^\infty(\Omega, \mathcal{F}, P), \quad x \mapsto L(x),$$

and let

$$\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$$

be a coherent risk measure. Further, let

$$\Phi_T,\ \Phi_I,\ \Phi_F : \mathbb{R} \longrightarrow [0,1]$$

be continuous functions generating, respectively, the truth-, indeterminacy-, and falsity-membership degrees. Then for each $x \in D$ *define*

$$T(x) = \Phi_T\big(\rho(L(x))\big), \quad I(x) = \Phi_I\big(\rho(L(x))\big), \quad F(x) = \Phi_F\big(\rho(L(x))\big).$$

The neutrosophic decision set *is*

$$\mathcal{N} = \big\{ \big(x,\ (T(x), I(x), F(x))\big) \mid x \in D \big\}.$$

Given nonnegative weights w_T, w_I, w_F *with* $w_T + w_I + w_F = 1$*, define the aggregated neutrosophic score*

$$S(x) = w_T\,T(x) \ +\ w_I\big(1 - I(x)\big) \ +\ w_F\big(1 - F(x)\big).$$

The neutrosophic risk management problem *is the optimization*

$$x^* \ \in\ \arg\max_{x \in D}\ S(x).$$

Example 2.17 (Neutrosophic CVaR Portfolio Selection)**.** *Consider a three-asset portfolio* $x = (w_1, w_2, w_3)$ *with* $w_i \geq 0$, $\sum_i w_i = 1$. *Model returns by* $S = 5$ *equiprobable scenarios* s*, each with probability* 0.2*, and let*

$$L_s(x) = -\sum_{i=1}^{3} w_i\,R_i^s,$$

so that $\rho(X) = \mathrm{CVaR}_{0.95}(X)$. *Choose*

$$\Phi_T(r) = e^{-10r}, \quad \Phi_I(r) = \tfrac{1}{2}\big(1 - e^{-10r}\big), \quad \Phi_F(r) = 1 - e^{-5r},$$

and weights $w_T = 0.5,\ w_I = 0.3,\ w_F = 0.2$. *Numerical solution yields* $\mathrm{CVaR}_{0.95}(L(x^*)) \approx 0.075$*, hence*

$$T(x^*) = e^{-0.75} \approx 0.472, \quad I(x^*) \approx 0.5\,(1 - e^{-0.75}) \approx 0.264, \quad F(x^*) = 1 - e^{-0.375} \approx 0.313,$$

and aggregated score

$$S(x^*) = 0.5 \cdot 0.472 + 0.3 \cdot (1 - 0.264) + 0.2 \cdot (1 - 0.313) \approx 0.687.$$

This example illustrates how truth, indeterminacy, and falsity degrees combine to guide selection.

Example 2.18 (Insurance Coverage Decision under Neutrosophic Risk). *Let $D = [0, 1]$ be the set of insurance coverage levels x, where $x = 1$ is full coverage and $x = 0$ no coverage. Loss given a claim is a random variable Y with*

$$P(Y = 1000) = 0.7, \quad P(Y = 10000) = 0.3.$$

Then for each $x \in D$,

$$L(x) = (1 - x)\, Y,$$

and we choose the risk measure $\rho(X) = \mathrm{CVaR}_{0.9}(X)$. Define neutrosophic membership generators

$$\Phi_T(r) = \exp(-r/1000), \quad \Phi_I(r) = \tfrac{1}{2}\big(1 - \exp(-r/1000)\big), \quad \Phi_F(r) = 1 - \exp(-r/2000),$$

and weights $w_T = 0.6,\ w_I = 0.2,\ w_F = 0.2$.

For $x = 0.8$ (80% coverage), the scenario losses are

$$L_1 = 0.2 \times 1000 = 200, \quad L_2 = 0.2 \times 10000 = 2000,$$

each with probabilities 0.7 and 0.3. The worst 10% of outcomes lies entirely in the L_2-scenario, so $\rho(L(0.8)) = 2000$. Hence

$$T(0.8) = \exp(-2) = 0.1353, \quad I(0.8) = \tfrac{1}{2}(1 - \exp(-2)) = 0.4323, \quad F(0.8) = 1 - \exp(-1) = 0.6321,$$

and the aggregated score is

$$S(0.8) = 0.6 \cdot 0.1353 \; + \; 0.2 \cdot (1 - 0.4323) \; + \; 0.2 \cdot (1 - 0.6321) \approx 0.0812 + 0.1135 + 0.0736 = 0.2683.$$

Example 2.19 (Inventory Order Quantity under Neutrosophic Risk). *Let $D = \{50, 100, 150\}$ be possible order quantities Q. Demand D has three equiprobable scenarios:*

$$P(D = 50) = P(D = 100) = P(D = 150) = \tfrac{1}{3}.$$

Holding cost is \$1 per unit leftover, shortage cost is \$5 per unit unmet demand. Thus

$$L(Q) = \begin{cases} (Q - D) \cdot 1, & Q \ge D, \\ (D - Q) \cdot 5, & Q < D. \end{cases}$$

We take $\rho(X) = \mathrm{CVaR}_{0.9}(X)$ again, and set

$$\Phi_T(r) = e^{-r/200}, \quad \Phi_I(r) = \tfrac{1}{2}\big(1 - e^{-r/200}\big), \quad \Phi_F(r) = 1 - e^{-r/400},$$

with weights $w_T = 0.5,\ w_I = 0.3,\ w_F = 0.2$.

For $Q = 100$:

$$L_1 = (100 - 50) \cdot 1 = 50, \quad L_2 = 0, \quad L_3 = (150 - 100) \cdot 5 = 250,$$

each with probability $1/3$. The worst 10% lies in the L_3-scenario, so $\rho(L(100)) = 250$. Then

$$T(100) = e^{-1.25} = 0.2865, \quad I(100) = \tfrac{1}{2}(1 - 0.2865) = 0.3568, \quad F(100) = 1 - e^{-0.625} = 0.4660,$$

and

$$S(100) = 0.5 \cdot 0.2865 \; + \; 0.3 \cdot (1 - 0.3568) \; + \; 0.2 \cdot (1 - 0.4660) \approx 0.1433 + 0.1930 + 0.1068 = 0.4431.$$

Theorem 2.20. *The neutrosophic framework strictly generalizes the fuzzy risk management model and endows the decision set with a neutrosophic structure. In particular, if one chooses*

$$\Phi_I(r) = 0, \quad \Phi_F(r) = 1 - \Phi_T(r), \quad w_T = 1,\ w_I = 0,\ w_F = 0,$$

then $\mathcal{N}$ *collapses to the fuzzy decision set* $\{\,(x, \Phi_T(\rho(L(x)))) \mid x \in D\}$ *and the neutrosophic problem reduces to* $\max_{x\in D} \Phi_T(\rho(L(x)))$.

Proof. Under the specializations $\Phi_I \equiv 0$ and $\Phi_F(r) = 1 - \Phi_T(r)$, the membership triple becomes $(T(x), 0, 1 - T(x))$. With $w_T = 1$ and $w_I = w_F = 0$, the aggregated score is

$$S(x) = T(x) = \Phi_T\big(\rho(L(x))\big),$$

recovering exactly the fuzzy-risk objective. Hence the neutrosophic model contains the fuzzy model as a special case while, for general Φ_I, Φ_F, it provides independent indeterminacy and falsity degrees, verifying the genuine extension. □

Theorem 2.21 (Monotonicity of Neutrosophic Score)**.** *Suppose* $\Phi_T\colon \mathbb{R} \to [0, 1]$ *is strictly decreasing,* $\Phi_I, \Phi_F\colon \mathbb{R} \to [0, 1]$ *are strictly increasing, and weights* $w_T, w_I, w_F \geq 0$ *satisfy* $w_T + w_I + w_F = 1$. *Then for any* $x, y \in D$,

$$\rho\big(L(x)\big) \leq \rho\big(L(y)\big) \quad \Longrightarrow \quad S(x) \geq S(y).$$

Proof. Let $r_x = \rho(L(x))$ and $r_y = \rho(L(y))$. If $r_x \leq r_y$, then

$$\Phi_T(r_x) \geq \Phi_T(r_y), \quad \Phi_I(r_x) \leq \Phi_I(r_y), \quad \Phi_F(r_x) \leq \Phi_F(r_y),$$

so

$$1 - \Phi_I(r_x) \geq 1 - \Phi_I(r_y), \quad 1 - \Phi_F(r_x) \geq 1 - \Phi_F(r_y).$$

Therefore

$$\begin{aligned} S(x) &= w_T\Phi_T(r_x) + w_I\big(1 - \Phi_I(r_x)\big) + w_F\big(1 - \Phi_F(r_x)\big) \\ &\geq w_T\Phi_T(r_y) + w_I\big(1 - \Phi_I(r_y)\big) + w_F\big(1 - \Phi_F(r_y)\big) = S(y). \end{aligned}$$

□

Theorem 2.22 (Boundedness)**.** *For all* $x \in D$,

$$0 \ \leq\ T(x),\ I(x),\ F(x)\ \leq\ 1, \quad 0\ \leq\ S(x)\ \leq\ 1.$$

Proof. By definition each Φ maps into $[0, 1]$, so $T, I, F \in [0, 1]$. Since S is a convex combination $S = w_T T + w_I(1 - I) + w_F(1 - F)$, and each term lies in $[0, 1]$, it follows $S(x) \in [0, 1]$. □

Theorem 2.23 (Continuity)**.** *If* $x \mapsto \rho(L(x))$ *is continuous on* D *and each* Φ_T, Φ_I, Φ_F *is continuous on* $\mathbb{R}$*, then the functions* $T, I, F\colon D \to [0, 1]$ *and* $S\colon D \to [0, 1]$ *are continuous.*

Proof. Each of $T = \Phi_T \circ (\rho \circ L)$, $I = \Phi_I \circ (\rho \circ L)$, and $F = \Phi_F \circ (\rho \circ L)$ is a composition of continuous maps, hence continuous. S is a weighted sum of these continuous functions and is therefore continuous. □

Theorem 2.24 (Existence of an Optimal Decision)**.** *If* $D \subset \mathbb{R}^n$ *is nonempty and compact, then there exists at least one* $x^* \in D$ *such that*

$$S(x^*)\ =\ \max_{x\in D} S(x).$$

Proof. By the Weierstrass extreme value theorem, any continuous real-valued function on a compact set attains its maximum. □

Theorem 2.25 (Lipschitz Continuity). *Assume $\rho \circ L$ is Lipschitz on D with constant L_ρ, and each Φ_T, Φ_I, Φ_F is Lipschitz on $\mathbb{R}$ with constant L_Φ. Then S is Lipschitz on D with constant $L_\Phi L_\rho$:*

$$\big|S(x) - S(y)\big| \;\leq\; L_\Phi\, L_\rho\, \|x - y\|, \quad \forall\, x, y \in D.$$

Proof. Write $r_x = \rho(L(x)), r_y = \rho(L(y))$. Then

$$\begin{aligned}|S(x) - S(y)| &\leq w_T|\Phi_T(r_x) - \Phi_T(r_y)| + w_I\,|\Phi_I(r_x) - \Phi_I(r_y)| + w_F\,|\Phi_F(r_x) - \Phi_F(r_y)| \\ &\leq (w_T + w_I + w_F)\, L_\Phi\, |r_x - r_y| = L_\Phi\, |r_x - r_y| \leq L_\Phi\, L_\rho\, \|x - y\|.\end{aligned}$$

□

3. Conclusion and Future Work

In this paper, we presented formally defined mathematical frameworks for Fuzzy Risk Management and Neutrosophic Risk Management, highlighting their theoretical foundations and key properties.

For future work, we plan to investigate further generalizations, including Hyperfuzzy Sets [48–52], Hesitant Fuzzy Sets[53, 54], Vague Sets[55, 56], Shadowed Sets[57–60], Quadri-Partitioned Neutrosophic Sets [61, 62], Double-Valued Neutrosophic Sets [63–66], Plithogenic Sets [67, 68], and Hyperneutrosophic Sets [69, 70]. We also anticipate advancing computational experiments and algorithm development to demonstrate practical applications of these models.

Conflicts of Interest

The authors declare no conflicts of interest in connection with this study or its publication.

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